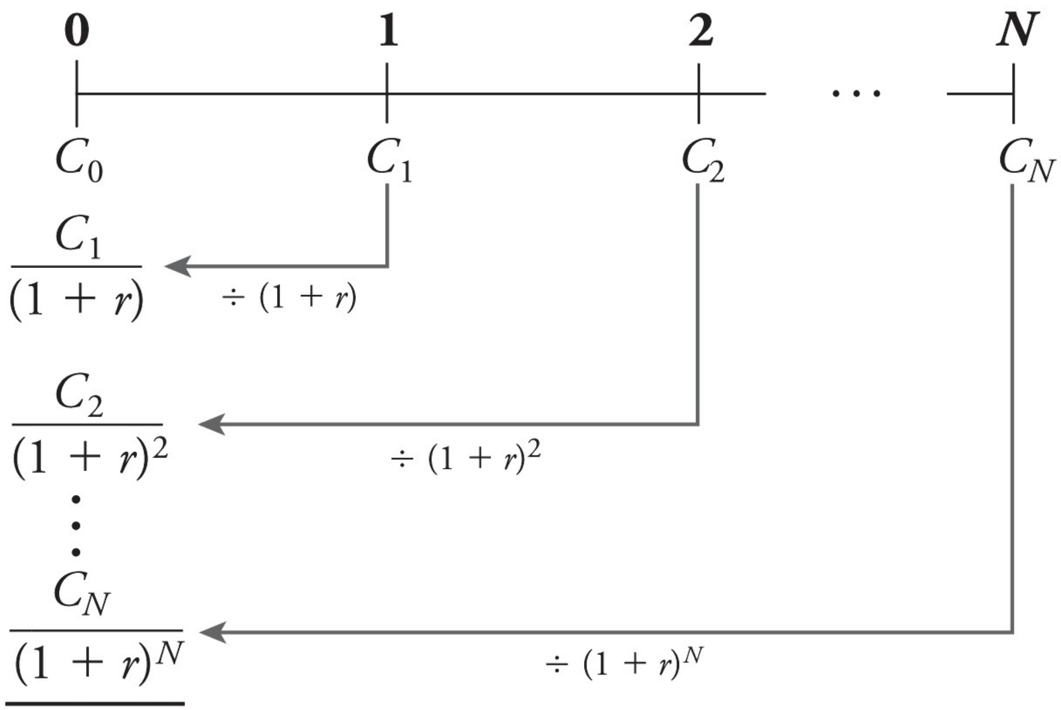
Business Finance : course n°5
Discounted Cash Flows (DCF) and Useful Shortcut
PV of an uneven cash flow stream
Annuities and perpetuities

BA 2nd Year - 01/10/2019

PV of an uneven cash flow stream

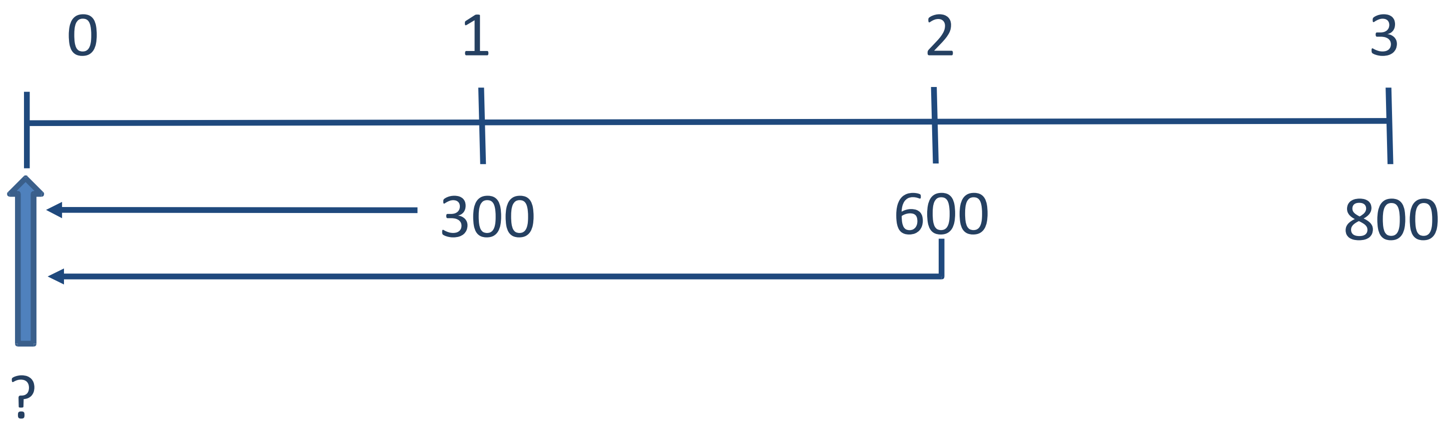


$$C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_N}{(1+r)^N}$$

$$PV = \sum_{n=0}^{N} PV(C_n) = \sum_{n=0}^{N} \frac{C_n}{(1+r)^n}$$

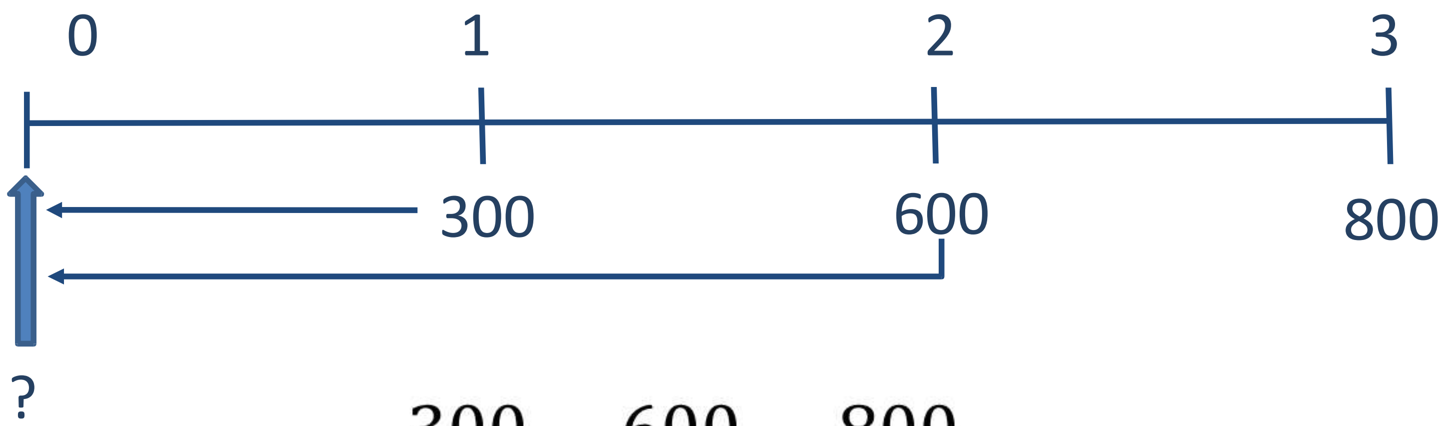
PV of an uneven cash flow stream

Practical work : find the PV of the following cash flow stream if the interest rate is 10% :



PV of an uneven cash flow stream

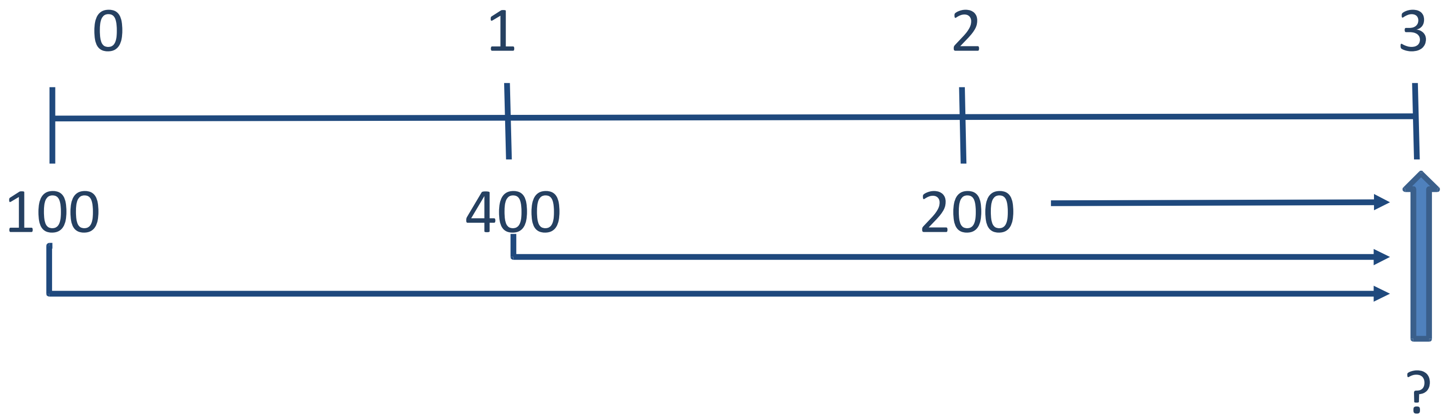
Find the PV of the following cash flow stream if the interest rate is 10%:



$$PV = \frac{300}{1.1} + \frac{600}{1.1^2} + \frac{800}{1.1^3} = \$\,1{,}369.65$$

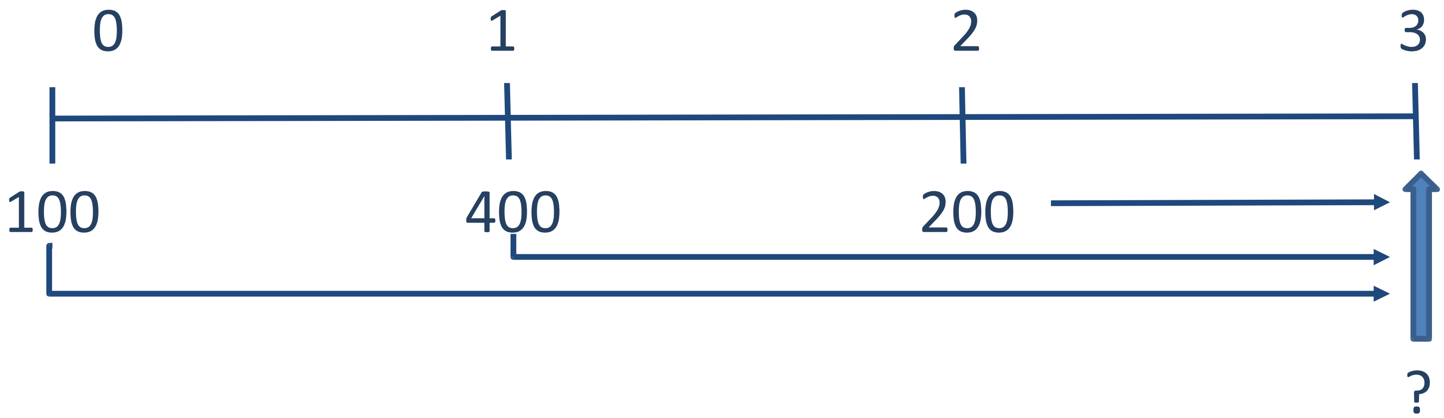
FV of an uneven cash flow stream

- Practical work : find the FV of the following cash flow stream if the interest rate is 10% :



FV of an uneven cash flow stream

- Find the FV of the following cash flow stream if the interest rate is 10%:



$$FV_3 = 100 \cdot 1.1^3 + 400 \cdot 1.1^2 + 200 \cdot 1.1$$
$$= \$\,837.1$$

Annuities and perpetuities

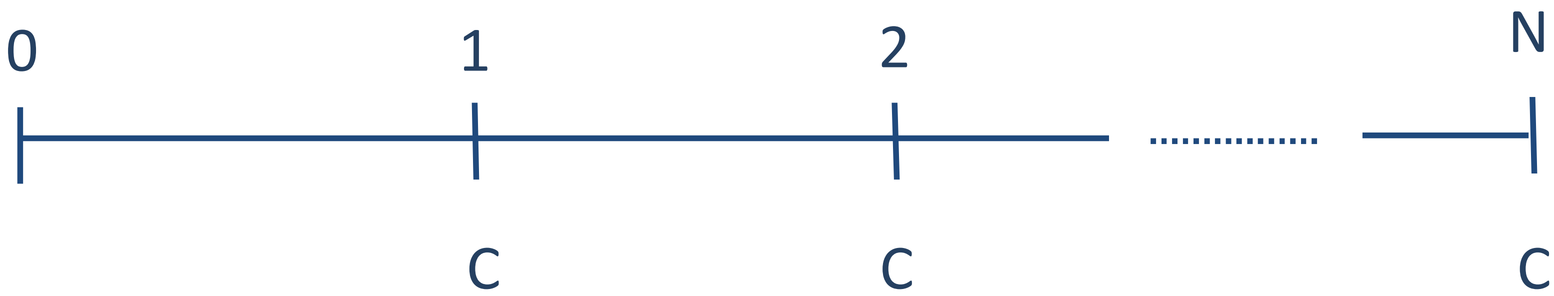
Cash flow streams: special cases

- Annuity
  - Ordinary annuity
  - Annuity due
- Perpetuity

Annuities

An **annuity** with maturity N years is a stream of N equal cash flows paid at regular intervals.

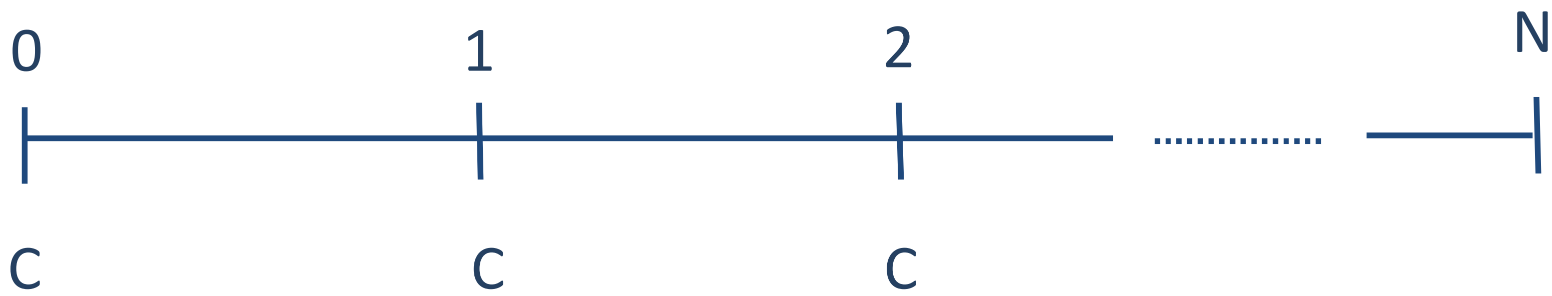
- **Ordinary annuity** with maturity N years and annual coupon C : cash flows are realized **at the end** of the year



Annuities

An **annuity** is a stream of N equal cash flows paid at regular intervals

- **Annuity due with maturity N years and annual coupon C** : cash flows are realized at the **beginning** of the year



PV of an ordinary annuity

- Practical work : What is the present value of an ordinary annuity with coupon $300 and maturity 3 years when the interest rate is 10%?

PV of an ordinary annuity

- What is the present value of an ordinary annuity with coupon $300 and maturity 3 years when the interest rate is 10%?

$$PV = \frac{300}{1.1} + \frac{300}{1.1^2} + \frac{300}{1.1^3} = \$\ 746.06$$

PV of an ordinary annuity

- Formula for the present value of an ordinary annuity, with annual payment C, maturity N years, when the interest rate is r.

$$\boldsymbol{PV = \frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right)}$$

- For the previous example, we find:

$$PV = \frac{300}{0.1}\left(1 - \frac{1}{1.1^3}\right) = \$\ 746.06$$

PV of an ordinary annuity : proof (very optional)

- The proof is derived in the same way as the sum of the elements of the geometric sequence

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} \dots + \frac{C}{(1+r)^N}$$

- Let $\frac{1}{1+r} = k$, then:

$$PV = Ck + Ck^2 + Ck^3 \dots + Ck^N \qquad (1)$$

- By multiplying *PV* by *k*, we get the following:

$$PV \cdot k = Ck^2 + Ck^3 + \cdots Ck^N + Ck^{N+1} \qquad (2)$$

- By subtracting (2) from (1), we get:

$$PV(1-k) = Ck(1-k^N)$$

- Since $\frac{1}{1+r} = k$:

$$PV\frac{r}{1+r} = C\frac{1}{1+r}\left(1 - \frac{1}{(1+r)^N}\right)$$

- Hence, $\boldsymbol{PV} = \frac{\boldsymbol{C}}{\boldsymbol{r}}\left(\mathbf{1} - \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{r})^{\boldsymbol{N}}}\right)$

FV of an ordinary annuity

- Practical work : What is the value in three years of an ordinary annuity with coupon \$300 and maturity 3 years when the interest rate is 10%?

FV of an ordinary annuity

- What is the value in three years of an ordinary annuity with coupon $300 and maturity 3 years when the interest rate is 10%?

- 1st method: direct computation:

$$FV = 300 + 300 \times 1.1 + 300 \times 1.1^2 = \$\,993$$

FV of an ordinary annuity

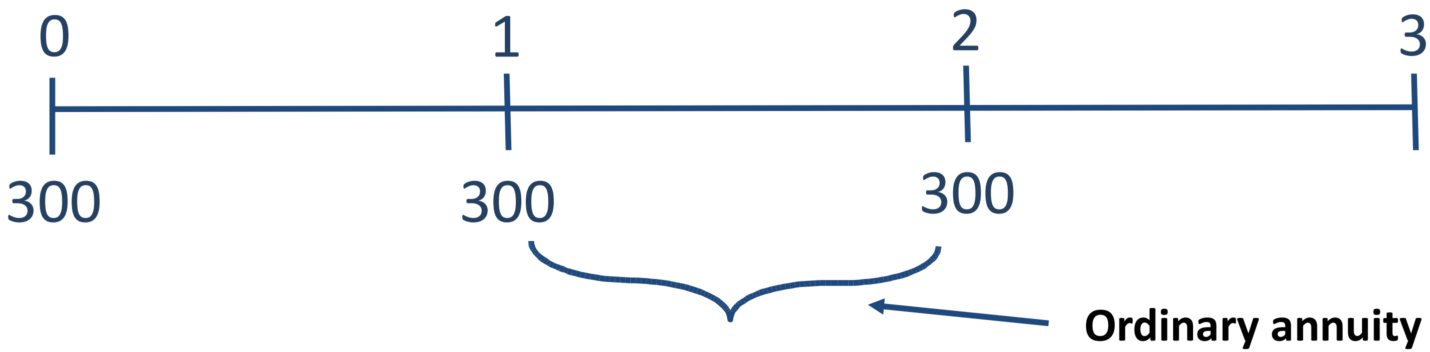
- 2nd method: using PV

$$FV = PV \times (1 + r)^n$$

$$FV = 746.06 \times 1.1^3 = \$\,993$$

PV of an annuity due

- What is the present value of an annuity due, with coupon $300 and maturity 3 years, if the interest rate is 10%?



- Remark: This 3-payment annuity due can be seen as a sum of today's cash flow and a 2-payment ordinary annuity.

PV of an annuity due

- First method: direct computation of PV:

$$PV = 300 + \frac{300}{1.1} + \frac{300}{1.1^2} = \$\ 820.66$$

- Second method: using the ordinary annuity

$$PV(annuity\ due, N\ payments) = C + PV(ordinary\ annuity, C, N-1\ payments)$$

$$PV = 300 + \frac{300}{0.1}\left(1 - \frac{1}{1.1^2}\right) = \$\ 820.66$$

Perpetuities

A perpetuity is a stream of equal cash flows that occur at regular intervals and last forever

The present value of perpetuity:

$$PV = \frac{C}{r}$$

PV of a perpetuity: proof (optional)

- The difference between perpetuity and annuity is that perpetuity is unlimited, while annuity ends at some future date N.
- Therefore, we can start by the formula for the PV of an annuity and modify it accordingly.
- PV of an ordinary annuity with N cash flows:

$$PV = \frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right)$$

- In the case of perpetuity: $N \to \infty$

Hence, $\lim_{N\to\infty} \frac{1}{(1+r)^N} = 0$ and $\lim_{N\to\infty} \frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right) = \frac{C}{r}$

- As a consequence the PV of a perpetuity is $PV = \frac{C}{r}$

PV of a perpetuity : example

- Practical work : since graduating from college, you have made a fortune. As a gesture of good will (with a tax benefit), you have decided to endow your alma mater with a research grant. The endowment implies a constant amount paid once a year in perpetuity. The first payment will be $20,000 and is to be made in exactly one year.
- If you are able to secure a 5% annual rate of return on the endowment fund, how much should you put into the endowment fund today?

PV of a perpetuity: example

- The timeline of the cash flows you want to provide is:

- This is a perpetuity of $20,000 annual payment. The funding you need to provide today is the present value of this perpetuity:

$$PV = \frac{20{,}000}{0{,}05} = \$\,400{,}000$$

- If you provide $400,000 today and if the endowment can earn a yield of 5% per year, then the college can provide an annual research grant of $20,000.

The Power of Compound Interest

- Practical work : a 20-year-old student wants to save $3 a day for her retirement. Every day she puts $3 in a drawer. At the end of the year, she invests the accumulated savings ($1,095) in a brokerage account with an expected annual return of 12%.

- How much money will she have when she is 65 years old?

The Power of Compound Interest

- The cash flows for this student correspond to an ordinary annuity, with annual payment \$1,095 and maturity 45 years.
- When she is 65, this person will get the future value of those cash flows:
- $FV = PV \times (1+r)^N = \frac{C}{r}\left(1 - \frac{1}{(1+r)^N}\right) \times (1+r)^N$
- FV=$\frac{1{,}095}{0.12}\left(1 - \frac{1}{(1.12)^{45}}\right) \times (1.12)^{45}$=\$ 1,487,262.89

Solving for **FV**: **If you don't start saving until you are 40** years old, how much will you have at 65?

If a 40-year-old investor begins saving today, and sticks to the plan, he or she will have $146,000.59 at age 65. This is $1.3 million less than if starting at age 20.

	BoP		12%	New Money
0				
1	$ 1 095,00	$	1 226,40	$ 1 095,00
2	$ 2 321,40	$	2 599,97	$ 1 095,00
3	$ 3 694,97	$	4 138,36	$ 1 095,00
4	$ 5 233,36	$	5 861,37	$ 1 095,00
5	$ 6 956,37	$	7 791,13	$ 1 095,00
6	$ 8 886,13	$	9 952,47	$ 1 095,00
7	$ 11 047,47	$	12 373,16	$ 1 095,00
8	$ 13 468,16	$	15 084,34	$ 1 095,00
9	$ 16 179,34	$	18 120,86	$ 1 095,00
10	$ 19 215,86	$	21 521,77	$ 1 095,00
11	$ 22 616,77	$	25 330,78	$ 1 095,00
12	$ 26 425,78	$	29 596,87	$ 1 095,00
13	$ 30 691,87	$	34 374,90	$ 1 095,00
14	$ 35 469,90	$	39 726,29	$ 1 095,00
15	$ 40 821,29	$	45 719,84	$ 1 095,00
16	$ 46 814,84	$	52 432,62	$ 1 095,00
17	$ 53 527,62	$	59 950,94	$ 1 095,00
18	$ 61 045,94	$	68 371,45	$ 1 095,00
19	$ 69 466,45	$	77 802,42	$ 1 095,00
20	$ 78 897,42	$	88 365,12	$ 1 095,00
21	$ 89 460,12	$	100 195,33	$ 1 095,00
22	$ 101 290,33	$	113 445,17	$ 1 095,00
23	$ 114 540,17	$	128 284,99	$ 1 095,00
24	$ 129 379,99	$	144 905,59	$ 1 095,00
25	$ 146 000,59	$	163 520,66	$ 1 095,00

How much must the 40-year old deposit annually to catch the 20-year old?

- To find the required annual contribution, remember that the final goal is to have $1,487,261.89 when she reach the retirement.

- This time, the maturity is 25 years and we have to find the annual payment C such that:

- 1,487,261.89$ = C $\frac{1}{0.12}\left(1-\frac{1}{(1+0.12)^{25}}\right)\times(1+0.12)^{25}$
- C = 1,487,261.89$/133.33 = 11,154.42$